⑥岩井班

Various Trigonometric Formulas Derived From The Infinite Product Representation of $\sin x$

Research group 6 (mathematics) "Learn from Euler"

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Introduction

Euler, an 18th-century Swiss mathematician, helped develop many functional equations by using infinite series and infinite products, including the solution of the Basel problem,$1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots=\frac{\pi^2}{6}$,which was an early achievement, and the results of which were later used in mathematics. That also had a great impact.

Infinite product of $\sin x$

Let $n=2m+1$, the factorization of x^n-1 can be expressed as

$$x^n-1=(x-1)\prod_{k=1}^{m}(x^2-2p_kx+1)\quad\left(p_k=\cos\frac{2\pi k}{n}\right).$$

We choose variables as follows $(a,\ b)\to(1+t,\ 1-t)$. We get

$$(1+t)^n-(1-t)^n=2nt\prod_{k=1}^{m}\left(1+t^2\cot^2\frac{\pi k}{n}\right).$$

In this equation, we replace t with x/n. So, we get

$$\left(1+\frac{x}{n}\right)^n-\left(1-\frac{x}{n}\right)^n=2x\prod_{k=1}^{m}\left(1+\frac{x^2}{n^2}\cot^2\frac{\pi k}{n}\right).$$

Here, considering the limit of $n\to\infty$, we get $\sinh x=x\prod_{k=1}^{\infty}\left(1+\frac{x^2}{\pi^2k^2}\right)$. We set $x\to ix,\ x\to\pi x$ and so on. We derive the equations, following:

$$\sin x=x\prod_{k=1}^{\infty}\left(1-\frac{x^2}{\pi^2k^2}\right),\quad\sinh\pi x=\pi x\prod_{k=1}^{\infty}\left(1+\frac{x^2}{k^2}\right),\quad\sin\pi x=\pi x\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2}\right).$$

Content

We get

$$\sin\pi x=\pi x\prod_{k=1}^{\infty}\left(1-\frac{x^2}{k^2}\right),\qquad(1+t)^n-(1-t)^n=2nt\prod_{k=1}^{m}\left(1+t^2\cot^2\frac{\pi k}{n}\right).$$

These two equations led to various relational expressions.

References

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